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An interactive boundary layer model compared to the triple deck theory

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Abstract

The aim of this article is to give a new insight leading to a better understanding of two-dimensional steady laminar incompressible separated flows on an indented flat plate. The asymptotic structure of the strong viscous interaction has been widely studied with the so-called "triple deck theory". This theory will be recalled firstly. For this, we will use the method of matched asymptotic expansions with a matching principle called "Modified Van Dyke principle" which removes all known counter-examples to the classical matching of Van Dyke. Then, using a new method called the "successive complementary expansions method", we are able to obtain the interactive boundary layer equations (IBL). The IBL theory relies upon generalized boundary layer equations which are strongly coupled to inviscid flow equations. Here, the IBL theory is established on a rational basis thanks to the use of *generalized* asymptotic expansions. It is then demonstrated that the triple deck is obtained as regular expansions of the IBL formulation.

Finally, numerical results for a standard indentation on the flat plate are given. IBL theory, contrary to triple deck, is non-local. Moreover, it is shown that the triple deck hypothesis of zero pressure gradient normal to the wall is not always appropriate.

1. Introduction

The first significant approach after Prandtl's original formulation of boundary layer theory was the Triple Deck Theory (TDT) attributed to Stewartson and Williams [1], Neiland [2] and Messiter [3]. The work of Stewartson and Williams was strongly influenced by an important contribution by Lighthill [4] who analyzed the upstream influence in supersonic flow. In TDT, a small perturbation, e.g. a hump, is placed on a flat plate generating an adverse pressure gradient, which can in turn provoke the separation of the flow.

Most numerical methods for solving the boundary-layer equations do not work when flow separation takes place. The classical feature of these methods is to impose the pressure gradient as a condition to solve the problem, the so-called direct approach.

This leads to a singularity, as shown by Goldstein [5], at the separation point, singularity that causes the breakdown of the numerical method. To remove the singularity, Catherall and Mangler [6] proposed the use of an inverse technique prescribing

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the displacement thickness as a boundary condition. The problem is that the displacement thickness is not known and must be obtained from the interaction between the inviscid flow and the boundary layer. In fact, all of these methods are based on separate treatments of the inviscid and viscous regions and can thus be classified in the *weak coupling* category. When we are concerned with separation or with the flow past trailing edges, the interaction is strong enough to impose a *strong coupling* between inviscid and viscous regions, a fact which requires the simultaneous treatment of the two zones. Veldman [7] has done this with a method prescribing a linear combination of pressure and displacement thickness. Until now, no rational arguments have been given for such a model.

The aim of this study is to unify some significant degeneracies of the Navier–Stokes equations in a single boundary layer model, which provides a uniformly valid approximation, including the normal pressure gradient. The key of the analysis is the use of generalized asymptotic expansions. This is a real improvement of what was done in the past as we shall see later.

In fact, it is well known that the interaction can be studied by means of asymptotic theories for large Reynolds numbers. In particular, laminar separated flows are treated by the so-called TDT. The strong coupling relating the outer deck with the viscous deck is issued from the asymptotic principle given by Van Dyke.

It is very interesting to note the thought process of Van Dyke:

"*Fortunately, since the two expansions have a common region of validity, it is easy to construct from them a single uniformly valid expansion.*"

In view of counterexamples showing that there is no overlap region, we think that the opposite is true: One must *first* assume the structure of a uniformly valid expansion and *then* infer a method of constructing this expansion. This could be the method of multiple scales or the WKB method. We shall use here an approach called the "Successive Complementary Expansions Method" (SCEM) which does not require the use of a matching principle [8]. In this way, we are able to construct a uniformly valid model leading to a boundary layer theory that includes the TDT as a special case. The strong coupling approach has been explored by Veldman and gives good results for the two-dimensional steady case. Afterwards, we will try to take into account the normal pressure gradient, a situation that reinforces the elliptic character of the model problem.

The paper is structured as follows: In Section 1, we will recall the triple deck theory. For this, we will use the method of matched asymptotic expansions (MMAE) with a matching principle called "Modified Van Dyke principle" (MVDP) which removes all known counter-examples to the classical matching of Van Dyke [9]. In addition, a set of equations containing the middle and the viscous layer is given. In Section 2, using the SCEM for Navier–Stokes equations at high Reynolds number [10], we obtain the interactive boundary layer (IBL) equations, capable of providing a non-zero normal pressure gradient. The IBL theory relies upon generalized boundary layer equations, which are strongly coupled to the inviscid flow equations. Here, the IBL theory is established on a rational basis thanks to the use of *generalized* asymptotic expansions. In Section 3, it is demonstrated that the TDT is obtained as regular expansions of the IBL formulation. Concerning this connection, see also [11]. In this section, it is also shown that the normal pressure gradient coming from the second order analysis of the TDT can be calculated from IBL. Finally, in Section 4, some numerical results for a standard indentation on the flat plate are given.

2. The triple deck theory from Navier–Stokes

We consider a two-dimensional incompressible flow on a flat plate with a uniform oncoming flow. The non-dimensional equations can be written,

$$\begin{aligned}
&v_x\frac{\partial v_x}{\partial x}+v_y\frac{\partial v_x}{\partial y}=-\frac{\partial \Pi}{\partial x}+\varepsilon^2\left(\frac{\partial^2 v_x}{\partial x^2}+\frac{\partial^2 v_x}{\partial y^2}\right),\\
&v_x\frac{\partial v_y}{\partial x}+v_y\frac{\partial v_y}{\partial y}=-\frac{\partial \Pi}{\partial y}+\varepsilon^2\left(\frac{\partial^2 v_y}{\partial x^2}+\frac{\partial^2 v_y}{\partial y^2}\right),\\
&\frac{\partial v_x}{\partial x}+\frac{\partial v_y}{\partial y}=0
\end{aligned}\tag{1}$$

where the Reynolds number is given by $\varepsilon^2 = Re^{-1}$. The coordinate along the wall is x and the coordinate normal to the wall is y. The x- and y-velocity components are v_x and v_y; the pressure is Π.

It is well known that, in TDT, we have to look first for a weak perturbation of a basic flow, which is (U_0, V_0) the classical Blasius solution, such as,

$$\begin{aligned}
&v_x(x,y,\varepsilon)=U_0(x,Y)+\varepsilon^{1/4}U(X,Y,\varepsilon)+\cdots,\\
&v_y(x,y,\varepsilon)=\varepsilon V_0(x,Y)+\varepsilon^{1/2}V(X,Y,\varepsilon)+\cdots,\\
&\Pi(x,y,\varepsilon)=\varepsilon^{1/2}P(X,Y,\varepsilon)+\cdots
\end{aligned}\tag{2}$$

where $X = (x - x_0)/\varepsilon^{3/4}$, x_0 being the location of a perturbation on the plate. If the length associated with x_0 is taken as reference length for the Reynolds number, then, $x_0 = 1$. With the boundary layer variable $Y = \frac{y}{\varepsilon}$, the continuity equation gives

$$U_{0x} + V_{0Y} = 0. \tag{3}$$

The Navier–Stokes equations gives the Blasius boundary layer

$$U_0 U_{0x} + V_0 U_{0Y} = U_{0YY}. \tag{4}$$

The equations for (U, V, P) can be written,

$$U_X + V_Y = 0, \tag{5a}$$

$$U_0 U_X + U_{0Y} V + \varepsilon^{1/4}(U U_X + V U_Y) = -\varepsilon^{1/4} P_X + \varepsilon^{3/4} U_{YY} + \mathrm{O}(\varepsilon^{3/4}), \tag{5b}$$

$$U_0 V_X + \varepsilon^{1/4}(U V_X + V V_Y) = -\varepsilon^{-1/4} P_Y + \varepsilon^{3/4} V_{YY} + \mathrm{O}(\varepsilon^{3/4}). \tag{5c}$$

With regular expansions, it is clear that the reduced equations are of first order. In order to fulfil the no-slip condition, we need a boundary layer variable,

$$\widetilde{Y} = \frac{y}{\varepsilon^{5/4}} = \frac{Y}{\varepsilon^{1/4}}. \tag{6}$$

In this viscous deck, we then look for approximations given by the regular expansions,

$$\begin{aligned} v_x(x, y, \varepsilon) &= U_0(x, Y) + \varepsilon^{1/4} \widetilde{U}_1(X, \widetilde{Y}) + \mathrm{O}(\varepsilon^{1/2}), \\ v_y(x, y, \varepsilon) &= \varepsilon V_0(x, Y) + \varepsilon^{3/4} \widetilde{V}_1(X, \widetilde{Y}) + \mathrm{O}(\varepsilon), \\ \Pi(x, y, \varepsilon) &= \varepsilon^{1/2} \tilde{P}_1(X, \widetilde{Y}) + \mathrm{O}(\varepsilon^{3/4}). \end{aligned} \tag{7}$$

Starting from (1) or (5) the first order equations are,

$$\widetilde{U}_{1X} + \widetilde{V}_{1\widetilde{Y}} = 0, \tag{8a}$$

$$\lambda_0(\widetilde{Y} \widetilde{U}_{1X} + \widetilde{V}_1) + (\widetilde{U}_1 \widetilde{U}_{1X} + \widetilde{V}_1 \widetilde{U}_{1\widetilde{Y}}) = -\widetilde{P}_{1X} + \widetilde{U}_{1\widetilde{Y}\widetilde{Y}}, \tag{8b}$$

$$\widetilde{P}_{1Y} = 0. \tag{8c}$$

We use the fact that the Blasius solution from (3) and (4) is given by the differential equation,

$$2f''' + ff'' = 0 \quad \text{where } U_0(x, Y) = f'(\eta) \text{ and } \eta = Y x^{-1/2}$$

$$\text{then, for } Y \to 0, \quad U_0(x, Y) \cong \lambda Y + \mathrm{O}(Y^4) \quad \text{with } \lambda = \lambda_0 x^{-1/2}.$$

For an indentation defined by $y = \varepsilon^{5/4} F((x - x_0)/\varepsilon^{3/4})$, the boundary conditions are,

$$\widetilde{U} = \widetilde{V} = 0 \quad \text{for } \widetilde{Y} = F(X).$$

The middle deck: **M**. From (5a–c) it is easily seen that regular expansions up to second order in the middle deck must be written,

$$\begin{aligned} U(X, Y, \varepsilon) &= U_1(X, Y) + \varepsilon^{1/4} U_2(X, Y) + \cdots, \\ V(X, Y, \varepsilon) &= V_1(X, Y) + \varepsilon^{1/4} V_2(X, Y) + \cdots, \\ P(X, Y, \varepsilon) &= P_1(X) + \varepsilon^{1/4} P_2(X, Y) + \cdots \end{aligned} \tag{9}$$

leading to the well-known first order solutions,

$$U_1(X, Y) = A_1(X) U_{0Y} \quad \text{and} \quad V_1(X, Y) = -A_{1X}(X) U_0, \tag{10}$$

where A_1 is an unknown first order displacement function.

The second order equations are,

$$\begin{aligned} &U_{2X} + V_{2Y} = 0, \\ &U_0 U_{2X} + U_{0Y} V_2 = -P_{1X} - (U_1 U_{1X} + V_1 U_{1Y}), \\ &U_0 V_{1X} = -P_{2Y}. \end{aligned} \tag{11}$$

Analytical solutions can easily be calculated, but as we are concerned with normal pressure gradient, we do not give them.

All these equations can be obtained from (5) as well as from Navier–Stokes equations. Thus Eqs. (5) can be considered as the basic model for both the middle and the viscous decks.

The matching **IM**. Now, the modified Van Dyke principle (MVDP) gives the matching conditions leading to the boundary conditions between the main deck **M** and the internal deck **I**.

To the order $O(\varepsilon^{1/4})$, we have $\mathbf{IM}v_x = \mathbf{MI}v_x$; this gives,

$$\lim_{\widetilde{Y}\to\infty} \widetilde{U}_1 = \lambda_0 A_1. \tag{12}$$

To the order $O(\varepsilon^{1/2})$, we have $\mathbf{IM}\Pi = \mathbf{MI}\Pi$; this gives,

$$P_1(X) = \widetilde{P}_1(X). \tag{13}$$

The matching conditions given above are derived by using the expansion operators **I** and **M**.

These operators are defined by regular expansions in the internal and the middle deck to the same order. It is reminded that the MVDP consists of applying operators **I** and **M** to the same order whereas the VDP can consider the same operators to any order.

The external deck: **E**. As V_1, from (10), does not vanish when $Y \to \infty$, we need an external deck given by the external variable which is known as,

$$Y^* = \frac{y}{\varepsilon^{3/4}} = Y\varepsilon^{1/4}. \tag{14}$$

We then look for the well-known following structure,

$$\begin{aligned} v_x(x,y,\varepsilon) &= 1 + \varepsilon^{1/2} U_1^*(X,Y^*) + \cdots, \\ v_y(x,y,\varepsilon) &= \varepsilon^{1/2} V_1^*(X,Y^*) + \cdots, \\ \Pi(x,y,\varepsilon) &= \varepsilon^{1/2} P_1^*(X,Y^*) + \cdots. \end{aligned} \tag{15}$$

This leads to the equations,

$$U_{1X}^* + V_{1Y^*}^* = 0, \quad U_{1X}^* = -P_{1X}^*, \quad V_{1X}^* = -P_{1Y^*}^* \tag{16}$$

which show that V_1^* and P_1^* are harmonic conjugate functions. The matching conditions **ME**, written to the order $O(\varepsilon^{1/2})$ for v_y and Π, give the boundary conditions for the first approximation of the external problem,

$$P_1^*(X,0) = P_1(X) = \widetilde{P}_1(X) \quad \text{and} \quad V_1^*(X,0) = -A_{1X}(X). \tag{17}$$

Conclusion. From (5), (16) and (17), the equations which contain the viscous and the middle layer can now be written

$$U_X + V_Y = 0, \tag{18a}$$

$$U_0 U_X + U_{0Y} V + \varepsilon^{1/4}(U U_X + V U_Y) = \varepsilon^{1/4} U_{1X0}^* + \varepsilon^{3/4} U_{YY}. \tag{18b}$$

The solution of the TD problem is given by Eqs. (18).

The matching condition on U is given by the PMVD: $\mathbf{ME}v_x = \mathbf{EM}v_x$ applied to the order $\varepsilon^{1/2}$.

One obtains,

$$\lim_{Y\to\infty} U = \varepsilon^{1/4} U_1^*(X,0) = \frac{\varepsilon^{1/4}}{\pi} \int_{-\infty}^{+\infty} \frac{V_1^*(\varsigma,0)}{X-\varsigma}\,\mathrm{d}\varsigma. \tag{19a}$$

The behaviour of the continuity equation (18a) at infinity gives,

$$\lim_{Y\to\infty} (V + \varepsilon^{1/4} Y U_{1X0}^*) = V_{10}^* \tag{19b}$$

where $U_{10}^* = U_1^*(X,0)$ and $V_{10}^* = V_1^*(X,0)$. Then, the normal pressure gradient P_Y can be calculated by,

$$P_Y = -\varepsilon^{1/4} U_0 V_X. \tag{20}$$

3. The interactive boundary layer

It is well known that the outer flow characteristics can be obtained from Navier–Stokes equations (1) by regular expansions such as,

$$\begin{aligned} v_x(x,y,\varepsilon) &= 1+\varepsilon u(x,y)+\cdots,\\ v_y(x,y,\varepsilon) &= \varepsilon v(x,y)+\cdots,\\ \Pi(x,y,\varepsilon) &= \varepsilon p(x,y)+\cdots. \end{aligned} \tag{21}$$

As a general rule, we will search for generalized asymptotic expansions beginning by,

$$\begin{aligned} v_x(x,y,\varepsilon) &= u_1(x,y,\varepsilon)+\cdots,\\ v_y(x,y,\varepsilon) &= v_1(x,y,\varepsilon)+\cdots,\\ \Pi(x,y,\varepsilon) &= p_1(x,y,\varepsilon)+\cdots. \end{aligned} \tag{22}$$

Formally neglecting terms of order ε^2, the triplet (u_1, v_1, p_1) satisfies the Euler equations,

$$\begin{aligned} &u_{1x}+v_{1y}=0,\\ &u_1u_{1x}+v_1u_{1y}=-p_{1x},\\ &u_1v_{1x}+v_1v_{1y}=-p_{1y} \end{aligned} \tag{23}$$

with, in our case, boundary conditions at infinity: $u_1 \to 1$ and $v_1 \to 0$. This is not a good approximation near the wall. A refinement is required. According to the SCEM, to complete (22) we are looking for a uniformly valid approximation (UVA) of the following form:

$$\begin{aligned} u_a(x,y,\varepsilon) &= u_1(x,y,\varepsilon)+U_1(x,Y,\varepsilon),\\ v_a(x,y,\varepsilon) &= v_1(x,y,\varepsilon)+\varepsilon V_1(x,Y,\varepsilon),\\ p_a(x,y,\varepsilon) &= p_1(x,y,\varepsilon)+\Delta(\varepsilon)P_1(x,Y,\varepsilon). \end{aligned} \tag{24}$$

The gauge function $\Delta(\varepsilon)$ is not known yet since the order of the pressure depends on the equations as we shall see. A point is that the normal pressure gradient is given by,

$$p_{ay} = p_{1y} + \frac{\Delta(\varepsilon)}{\varepsilon}P_{1Y}. \tag{25}$$

Taking into account Eqs. (23) and neglecting terms $\mathrm{O}(\varepsilon^2)$, Navier–Stokes and continuity equations can now be written,

$$\begin{aligned} &U_{1x}+V_{1Y}=0,\\ &(u_1+U_1)U_{1x}+\left(\frac{v_1}{\varepsilon}+V_1\right)U_{1Y}+U_1u_{1x}+\varepsilon V_1u_{1y}=-\Delta P_{1x}+U_{1YY},\\ &(u_1+U_1)\varepsilon V_{1x}+(v_1+\varepsilon V_1)V_{1Y}+U_1v_{1x}+\varepsilon V_1v_{1y}=-\frac{\Delta}{\varepsilon}P_{1Y}+\varepsilon V_{1YY}. \end{aligned}$$

In the last equation, as well as all other terms, v_1 is of order ε in the boundary layer. This means that a good choice is $\Delta(\varepsilon)=\varepsilon^2$. Using (23) and (24), these equations can be written,

$$\begin{aligned} &u_{ax}+v_{ay}=0,\\ &u_au_{ax}+v_au_{ay}=-p_{1x}+\varepsilon^2(u_{ayy}-u_{1yy}),\\ &u_av_{ax}+v_av_{ay}=-p_{ay}+\varepsilon^2(v_{ayy}-v_{1yy}). \end{aligned} \tag{26}$$

Solutions of Eqs. (23) and (26) give a uniformly valid approximation over the whole domain and not only in the boundary layer. In order to compare existing models to the one described by Eqs. (26), suitable manipulations can be done as follows. From (24), in the boundary layer where Y is of order 1 and since $y=\varepsilon Y$, we can write,

$$u_a = u_{10} + U_1 + \cdots,$$
$$v_a = v_{10} - y u_{1x0} + \varepsilon V_1 + \cdots,$$
$$p_{ay} = p_{1y0} + y p_{1yy0} + \varepsilon P_{1Y} + \cdots$$

where, $u_{10} = u_1(x,0,\varepsilon)$, $v_{10} = v_1(x,0,\varepsilon)$, $u_{1x0} = u_{1x}(x,0,\varepsilon)$, $p_{1y0} = p_{1y}(x,0,\varepsilon)$, $p_{1yy0} = p_{1yy}(x,0,\varepsilon)$. This suggests defining:

$$U = u_{10} + U_1, \quad \varepsilon V = v_{10} - y u_{1x0} + \varepsilon V_1, \quad \varepsilon P_Y = p_{1y0} + y p_{1yy0} + \varepsilon P_{1Y}.$$

This leads to a new form of the UVA:

$$\begin{aligned} u_a &= U + u_1 - u_{10}, \\ v_a &= \varepsilon V + v_1 - v_{10} + y u_{1x0}, \\ p_{ay} &= \varepsilon P_Y + p_{1y} - p_{1y0} - y p_{1yy0}. \end{aligned} \tag{27}$$

As boundary conditions are, for $Y \to \infty$, $U_1 \to 0$, $V_1 \to 0$, we can write,

$$\lim_{Y\to\infty} U = u_{10}, \qquad \lim_{Y\to\infty} (V + Y u_{1x0}) = \frac{v_{10}}{\varepsilon}. \tag{28}$$

The second condition indicates the behavior of V at infinity; this is, as we shall see, the origin of the coupling relation in TDT. At the wall, we simply have, $U = V = 0$. It is easy to see that, if the flow governed by the Euler equations is irrotationnal, then, in the boundary layer, we have,

$$u_a = U + \mathrm{O}(\varepsilon^2), \quad v_a = \varepsilon V + \mathrm{O}(\varepsilon^3), \quad p_{ay} = \varepsilon P_Y + \mathrm{O}(\varepsilon^2).$$

To the same order $\mathrm{O}(\varepsilon^2)$, Eqs. (26) can now be written in the boundary layer,

$$U U_x + V U_Y = u_{10} u_{1x0} + U_{YY}, \tag{29a}$$
$$U V_x + V V_Y = -P_Y + V_{YY}, \tag{29b}$$
$$U_x + V_Y = 0. \tag{29c}$$

It is easily seen that boundary conditions at infinity (28) give the needed behavior for these equations. Nevertheless, as U and V can be calculated independently of the pressure, the second equation gives the normal pressure gradient. The first equation is nothing else than the standard Prandtl's equation used by Veldman in his calculations. This model, called "generalized Prandtl's model", here fully justified, has to be solved with the Euler equations. To summarize, the uniformly valid approximation is obtained by solving Eqs. (23) and (29) with conditions (28), no slip conditions at the wall and uniform flow conditions at infinity. It is emphasized that no matching conditions are used to obtain this model. It has been assumed simply that the form of the UVA is given by (24) with $\Delta(\varepsilon) = \varepsilon^2$.

4. The triple deck theory from IBL

Here, we will show that the IBL model (29) contains the triple deck approximations. The scales could be deduced from the model but, for the sake of simplicity, it is assumed that they are known. For instance, TD is a theory written in the vicinity of a point x_0 such as, $X = (x - x_0)/\varepsilon^{3/4}$. In this sense, the triple deck is a local theory.

The outer deck. In the outer deck, from (14), the appropriate variable normal to the wall is Y^*. Moreover, from (15) we have to write the following normalization,

$$u_1 = 1 + \varepsilon^{1/2} U^*, \quad v_1 = \varepsilon^{1/2} V^*, \quad p_1 = \varepsilon^{1/2} P^*.$$

This leads, from (23), to the outer equations,

$$\begin{aligned} &U^*_X + V^*_{Y^*} = 0, \\ &U^*_X = -P^*_X, \\ &V^*_X = -P^*_{Y^*}. \end{aligned} \tag{30}$$

We obtain the same equations (16) and (19) as for the TDT. The UVA, from (27), takes the form,

$$
\begin{aligned}
u_a &= U + \varepsilon^{1/2}(U^* - U_0^*),\\
v_a &= \varepsilon^{1/2}(V^* - V_0^* + Y^* U_{X0}^*) + \varepsilon V,\\
p_{ay} &= \varepsilon^{-1/4}(P_{Y^*}^* - P_{Y^*0}^* - Y^* P_{Y^*Y^*0}^*) + \varepsilon P_Y
\end{aligned}
\tag{31}
$$

with, $U_0^* = U^*(X, 0, \varepsilon)$, $V_0^* = V^*(X, 0, \varepsilon)$, $U_{X0}^* = U_X^*(X, 0, \varepsilon)$, $P_{Y^*0}^* = P_{Y^*}^*(X, 0, \varepsilon)$ and $P_{Y^*Y^*0}^* = P_{Y^*Y^*}^*(X, 0, \varepsilon)$. The boundary conditions for U and V must now be written,

$$
\lim_{Y\to\infty} U = 1 + \varepsilon^{1/2} U_0^*, \qquad \lim_{Y\to\infty} (V + \varepsilon^{-1/4} Y U_{X0}^*) = \varepsilon^{-1/2} V_0^*. \tag{32}
$$

The main and inner decks. As suggested by (2), in (31), U, V and P are written in the following form,

$$
\begin{aligned}
U(x, Y, \varepsilon) &= U_0(x, Y) + \varepsilon^{1/4} \widehat{U}(X, Y, \varepsilon),\\
V(x, Y, \varepsilon) &= V_0(x, Y) + \varepsilon^{-1/2} \widehat{V}(X, Y, \varepsilon),\\
P(x, Y, \varepsilon) &= \varepsilon^{-3/2} \widehat{P}(X, Y, \varepsilon).
\end{aligned}
$$

Eqs. (29) give,

$$
\widehat{U}_X + \widehat{V}_Y = 0, \tag{33a}
$$

$$
U_0 \widehat{U}_X + U_{0Y} \widehat{V} + \varepsilon^{1/4}(\widehat{U}\widehat{U}_X + \widehat{V}\widehat{U}_Y) = \varepsilon^{1/4} U_{X0}^* + \varepsilon^{3/4} \widehat{U}_{YY} + \mathrm{O}(\varepsilon^{3/4}), \tag{33b}
$$

$$
\varepsilon^{1/4} U_0 \widehat{V}_X = -\widehat{P}_Y + \mathrm{O}(\varepsilon^{1/2}). \tag{33c}
$$

The uniformly valid approximation (31) is now looked for as,

$$
\begin{aligned}
u_a &= U_0 + \varepsilon^{1/4} \widehat{U} + \varepsilon^{1/2}(U^* - U_0^*),\\
v_a &= \varepsilon^{1/2}(\widehat{V} + V^* - V_0^* + Y^* U_{X0}^*) + \varepsilon V_0,\\
p_{ay} &= \varepsilon^{-1/4}(P_{Y^*}^* - P_{Y^*0}^* - Y^* P_{Y^*Y^*0}^*) + \varepsilon^{-1/2}\, \widehat{P}_Y\ .
\end{aligned}
\tag{34}
$$

From (32), the boundary conditions give, at the order $\varepsilon^{1/4}$,

$$
\lim_{Y\to\infty} \widehat{U} = \varepsilon^{1/4} U_{10}^*, \qquad \lim_{Y\to\infty} (\widehat{V} + \varepsilon^{1/4} Y U_{X0}^*) = V_{10}^*. \tag{35}
$$

Now, we have shown that the outer equations are the same. Moreover, Eqs. (33) and (18) give the same result in the middle layer and the the viscous layer. The first condition (35) giving in fact the same result as (19), the triple deck theory is contained in the theory of interacting boundary layer with the calculation of the normal pressure gradient.

5. Numerical calculations

In order to illustrate the IBL model, numerical computations have been performed for boundary layer flow over a flat plate with and without an indentation. We will focus on the normal pressure gradient for a two-dimensional, steady, laminar and incompressible flow. The IBL model considered is given by the $\mathrm{O}(\varepsilon^2)$ set of Eqs. (29). As mentioned before, the second IBL equation (29b) is uncoupled from the other two equations, so that solving the normal pressure gradient is straightforward provided the velocity field is known. Eqs. (29a) and (29c) have to be simultaneously solved with the Euler equations (23). As for the case of Blasius flow, we have to write,

$$
u_1(x, y, \varepsilon) = 1 + \varepsilon u(x, y), \quad v_1(x, y, \varepsilon) = \varepsilon v(x, y), \quad p_1(x, y, \varepsilon) = \varepsilon p(x, y).
$$

From (27) and (28), the boundary conditions are:

$$
U = V = 0 \quad \text{for } Y = \varepsilon f(x),
$$

$$
\lim_{Y\to\infty} U = 1 + \varepsilon u_0, \qquad \lim_{Y\to\infty} (V + \varepsilon Y u_{0x}) = v_0
$$

where, $u_0 = u(x, 0)$ and $v_0 = v(x, 0)$. Under these assumptions, the IBL model leads to the model used by Veldman [7]. In the following we have used a quasi-simultaneous method to solve boundary layer flows.

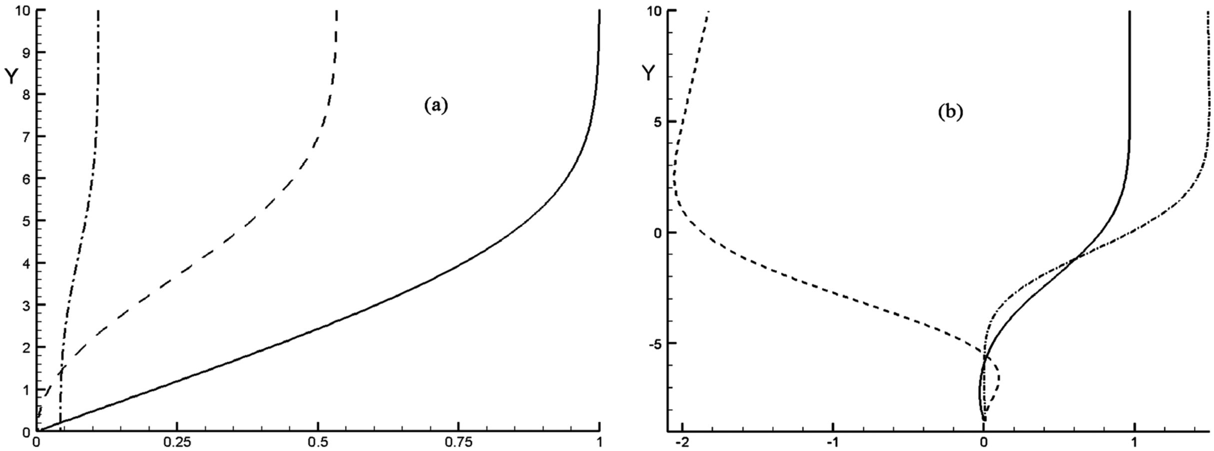

Fig. 1. (a) and (b): profiles of streamwise velocity U (solid line), transverse velocity V (dashed line) and normal pressure gradient P_Y (dash-dotted line) versus the normal boundary layer co-ordinate Y at $x = 2.5$ and for $Re = 80\,000$. (a) the flat plate. (b) the Carter and Wornom indented plate, for this case P_Y has been divided by 100.

The first case considered is the flat plate ($f(x) = 0$) for $Re = 80\,000$. Fig. 1(a) shows the streamwise and transverse velocity, and normal pressure gradient profiles at $x = 2.5$. This latter ranges approximately from 0.04 to 0.1.

The second case investigated is the Carter and Wornom indented plate also considered by Veldman, i.e. the equation of the plate is $f(x) = -0.03\,\mathrm{sech}\,4(x - 2.5)$, the Reynolds number is left unchanged. Profiles are shown on Fig. 1(b) at the same abscissa as for the flat plate, i.e. at the deepest point of the trough. The magnitude of the transverse pressure gradient has been divided by 100 in order to fit the same graph with other profiles. Profiles have been truncated at $Y = 10$, above the P_Y profile is quite unchanged up to the computation domain boundary. Calculations of P_Y have also been made with the TD result (20). On Fig. 1(b) it gives the same result as IBL. For this particular case, the separation bubble upper limit is near $Y = -4.75$ at $x = 2.5$. It can be seen that in the bubble, the transverse pressure gradient is close to zero. This feature is consistent with the TDT, for which this zone belongs to the viscous sub-layer where $P_Y = 0$. More important is the fact that the maximum value of P_Y is about 150 whereas for the flat plate it is about 0.2, thus P_Y is about 3 orders of magnitude higher when separation occurs. Any normal pressure gradient profile taken in the separation bubble area has roughly the same order of magnitude as the one considered above. This is not the case outside this area, the profiles magnitude quickly diminishes. In view of $p_{ay} = p_{1y} + \varepsilon P_{1Y} = \varepsilon(p_y + P_Y)$, the maximum of p_{ay} is about 0.5, which is clearly O(1). Then the hypothesis of negligible normal pressure gradient is not obvious when separation occurs.

6. Concluding remarks

The basis of the analysis of high Reynolds numbers flows near walls is the interactive boundary layer (IBL) theory which ensures a strong coupling between the boundary layer and the inviscid equations which rule the outer motion.

It is first shown that the triple deck equations can be established by employing a modified Van Dyke principle (MVDP), with the method of matched asymptotic expansions (MMAE). It must be pointed out that MMAE is based on regular expansions, and this is the reason why triple decks are necessary to ensure strong coupling.

Secondly, a sound justification of IBL has been provided, which includes the calculation of the normal pressure gradient. This result is directly associated with the determination of uniformly valid approximations which are obtained by the use of *generalized* asymptotic expansions, the so-called "successive complementary expansion method" (SCEM). This approach may be formulated with different levels of complexity, one of them leading to, what we can call, the "generalized Prandtl's equations". Moreover, the conditions which must be used to solve the IBL do not result from any matching principle.

Thirdly, it has been shown that the triple deck theory (TDT) is deduced from IBL as regular asymptotic expansions. Finally, numerical calculations show that, when separation occurs, the hypothesis of a negligible normal pressure gradient in the boundary layer is not tenable. This point needs further analysis in the frame of the SCEM, i.e. uniformly valid asymptotic approximations must be built with the requirement that both velocity and pressure exhibit a dependency on the normal to the wall direction.

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